ENGR 76

Information Science and Engineering

Lecture 16: Channel Capacity & Concluding Remarks

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Binary Symmetric Channel

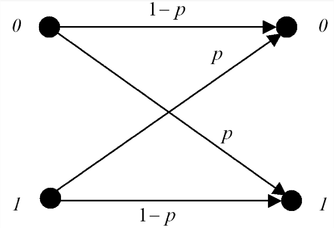
Conditional Probability

- $P(B|A)$
  - Probability of event B happening given that event A has happened
  - Assume $P(A) > 0$

$$P(B|A) = \frac{P(B, A)}{P(A)}$$

  - Intuition: Fraction of times both A and B happen divided by fraction of times A happens

Binary Symmetric Channel (BSC)



- Transmitted bit X
- Received bit Y
- Received correctly with probability $1-p$, flipped with probability p

$$P(Y=0|X=0)=1-p \quad P(Y=1|X=0)=p$$
$$P(Y=1|X=1)=1-p \quad P(Y=0|X=1)=p$$

Memoryless BSC

- Each output bit depends only on the corresponding input bit
- **Bit flips are independent across channel uses**
  - Each bit is flipped (or not flipped) independently of the other bits

Bit Error Rate

- Information bit b
- Decoded information bit $\hat{b}$
- Bit error rate is the probability of incorrect decoding, i.e.,

$$BER = P(\hat{b} \neq b)$$

Uncoded Transmission

- Transmit the bit b directly
- What is the BER?

Uncoded Transmission

- Transmit the bit b directly
- What is the BER?

$$P(\hat{b} = 1|b = 0) = P(\hat{b} = 0|b = 1) = p$$

- Bit error rate is p

Repetition Coding

- Code: $0 \to 000 \qquad 1 \to 111$
- Decoding: minimum distance decoding
  - Decode $000, 100, 010, 001$ as $\hat{b} = 0$
  - Decode $111, 011, 101, 110$ as $\hat{b} = 1$
- $P(\hat{b} = 1 | b = 0) = ?$

Repetition Coding

$$
\begin{aligned}
&P(\hat{b} = 1 | b = 0) \\
&= P(\text{Received seqn is } 111 \text{ or } 011 \text{ or } 101 \text{ or } 110 \,\big|\, \text{Transmitted seqn is } 000) \\
&= P(\text{receive} = 111 | \text{transmit} = 000) + P(\text{receive} = 011 | \text{transmit} = 000) \\
&\quad + P(\text{receive} = 101 | \text{transmit} = 000) + P(\text{receive} = 110 | \text{transmit} = 000) \\
&= p^3 + 3p^2(1 - p)
\end{aligned}
$$

Repetition Coding

- If p is much smaller than 1 (e.g., $p = 0.01$), then

$$\begin{aligned} BER &= 3p^2(1-p) + p^3 \\ &\approx 3p^2 \end{aligned}$$

- For $p = 0.01$, BER reduces from 0.01 to 0.0003
- But transmission rate reduces to $\frac{1}{3}$

Repetition Coding with $d_{min} = 5$

- Code: $0 \rightarrow 00000 \qquad 1 \rightarrow 11111$
- Error in decoding: when at least **three** bits are flipped
  - Transmitted 00000 and received 11100
- $BER \propto p^3$
- Transmission Rate $R = \frac{1}{5}$

Tradeoff?

Code	BER	Rate
Uncoded Transmission	p	1
Repetition Code ($d_{\min} = 3$)	p^2	$\frac{1}{3}$
Repetition Code ($d_{\min} = 5$)	p^3	$\frac{1}{5}$

- Seems to be a tradeoff:
    - Lower BER requires a lower rate
- **Is this tradeoff fundamental?**

Shannon's Channel Coding Theorem

Theorem (Shannon (1948))

For every communication channel, there exists a $C > 0$ (information bits/channel use) called the **channel capacity** such that:

- If the transmission rate $R < C$, then there exist codes whose probability of error can be made arbitrarily small, provided the block length is large enough.
- For any transmission rate $R > C$, reliable communication is impossible, i.e., errors are inevitable.

- C only depends on the channel and can be computed from the conditional distribution of the output given the input for the channel

Capacity Achieving Codes

What code achieves rate close to C while allowing for error probability to be arbitrarily small?

- Shannon's proof only showed existence
  - Uses randomly constructed codes
  - Computationally impractical encoding and decoding
- Codes studied in class are not capacity achieving
  - Convolutional Codes: BER scales roughly as p^3
  - Error probability does not vanish as block length increases
  - Still very useful
- Remained an open problem for more than 50 years

Capacity Achieving Codes

What code achieves rate close to C while allowing for error probability to be arbitrarily small?

- **Polar Codes** (2008)
  - First explicit code proven to achieve capacity
  - Developed by Erdal Arikan
  - Practical encoding/decoding complexity

Closing the Loop

Closing the Loop

- Started with a bunch of questions
- Let's see how many of them we can answer now

- What is information and how do we measure it?
- How is information represented using bits?
- How to efficiently store information and what makes compression possible?
- What is the frequency domain representation and how is it used?
- How to convert bits to physical waveforms?
- How do we manage reliable communication in a noisy environment?

What's Next?

- All topics that we covered have courses dedicated to them

Information Theory and Data Compression

- EE 276: Information Theory
  - Theory behind source coding, channel coding, and information measures
  - Foundations used in communications, machine learning, and statistics
- EE 274: Data Compression, Theory and Applications
  - Both fundamental and modern compression algorithms
  - Covers very recent research: algorithms developed over the last few years

Signal Processing and Fourier Transform

- EE 102A/102B: Signal Processing I and II
- EE 261: Fourier Transform and its Applications
  - Fundamentals behind frequency-domain analysis
  - Applications in signal processing, imaging, and communications
- EE 264: Digital Signal Processing

Communication

- EE 179/279/379: Analog and Digital Communication
  - AM, FM, and modern digital modulation schemes
  - Design and analysis of practical communication systems

Error Correcting Codes

- CS 250/EE 387: Algebraic Error Correcting Codes
- EE 388: Modern Coding Theory

Thank You!

- Hope you enjoyed the course!
- Please remember to submit course feedback
    - This offering of the course was different
    - Your feedback will be especially helpful

Thank You!